

FG-coupled fixed point theorems for contractive and generalized quasi-contractive mappings

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Abstract

In this paper we prove FG-coupled fixed point theorems for different contractive mappings and generalized quasi-contractive mappings in partially ordered complete metric spaces. We prove the existence of FG-coupled fixed points of continuous as well as discontinuous mappings. We give some examples to illustrate the results.

Key words: FG-coupled fixed point, mixed monotone property, partially ordered complete metric space, contractive mapping, quasi-contractive mapping.

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1 Introduction and Preliminaries

Coupled fixed point problems become a new trend in non-linear analysis as a generalization of fixed point theory. In 1987 Dajun Guo and V. Lakshmikantham [6] introduced the concept of coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings in cone metric spaces. Later in 2006 Gnana Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point and mixed monotone property for contractive mappings in partially ordered complete metric spaces. Thereafter by changing the spaces and using different contractions several authors have

proved various coupled fixed point theorems [1, 2, 7, 9].

Recently E. Prajisha and P. Shaini [8] defined FG-coupled fixed point on product of two spaces as a generalization of coupled fixed point and some FG-fixed point theorems have been proved. In this paper we prove FG- coupled fixed point theorems for contractive type mappings on partially ordered complete metric spaces. Our results generalizes several fixed point theorems in literature. Now we recall some definitions.

Definition 1.1 ([5]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the map $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2 ([8]) Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered metric spaces and $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$. We say that F and G have mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \leq_{P_1} x_2 \Rightarrow F(x_1, y) \leq_{P_1} F(x_2, y) \text{ and } G(y, x_1) \geq_{P_2} G(y, x_2)$$

$$y_1, y_2 \in Y, \quad y_1 \leq_{P_2} y_2 \Rightarrow F(x, y_1) \geq_{P_1} F(x, y_2) \text{ and } G(y_1, x) \leq_{P_2} G(y_2, x).$$

Definition 1.3 ([8]) An element $(x, y) \in X \times Y$ is said to be FG- coupled fixed point if $F(x, y) = x$ and $G(y, x) = y$.

Notes:

- 1) If $(x, y) \in X \times Y$ is an FG- coupled fixed point then $(y, x) \in Y \times X$ is GF- coupled fixed point.
- 2) The metric d on $X \times Y$ is defined by $d((x, y), (u, v)) = d_X(x, u) + d_Y(y, v)$ for all $(x, y), (u, v) \in X \times Y$.
- 3) Partial order \leq on $X \times Y$ is defined by for any $(x, y), (u, v) \in X \times Y$; $(u, v) \leq (x, y) \Leftrightarrow x \geq_{P_1} u, y \leq_{P_2} v$.
- 4) $F^{n+1}(x, y) = F(F^n(x, y), G^n(y, x))$ and $G^{n+1}(y, x) = G(G^n(y, x), F^n(x, y))$ for every $n \in \mathbb{N}$ and $(x, y) \in X \times Y$.

2 FG-coupled fixed point theorems for contraction mappings

Theorem 2.1 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two continuous mappings having the mixed monotone property. Assume that there exist $k, l, m, n \in [0, 1]; k + l < 1$ and $m + n < 1$ with*

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, u) + l d_Y(y, v) \quad \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (1)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, v) + n d_X(x, u) \quad \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (2)$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exist $(x, y) \in X \times Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Given $x_0 \leq_{P_1} F(x_0, y_0) = x_1$ (say) and $y_0 \geq_{P_2} G(y_0, x_0) = y_1$ (say).

Define $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(y_n, x_n)$ for $n = 1, 2, 3, \dots$

Then we get $F^{n+1}(x_0, y_0) = x_{n+1}$ and $G^{n+1}(y_0, x_0) = y_{n+1}$.

Using mathematical induction and mixed monotone property of F and G we prove that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y. For,

Given $x_0 \leq_{P_1} x_1$ and $y_0 \geq_{P_2} y_1$. Claim that $x_n \leq_{P_1} x_{n+1}$ and $y_n \geq_{P_2} y_{n+1} \quad \forall n \in \mathbb{N}$.

For $n = 1$, $x_2 = F(x_1, y_1) \geq_{P_1} F(x_0, y_1) \geq_{P_1} F(x_0, y_0) = x_1$ and

$y_2 = G(y_1, x_1) \leq_{P_2} G(y_1, x_0) \leq_{P_2} G(y_0, x_0) = y_1$.

Assume that the result is true for $n=m$. ie, $x_{m+1} \geq_{P_1} x_m$ and $y_m \geq_{P_2} y_{m+1}$.

Now consider,

$$x_{m+2} = F(x_{m+1}, y_{m+1}) \geq_{P_1} F(x_m, y_{m+1}) \geq_{P_1} F(x_m, y_m) = x_{m+1}$$

$$y_{m+2} = G(y_{m+1}, x_{m+1}) \leq_{P_2} G(y_{m+1}, x_m) \leq_{P_2} G(y_m, x_m) = y_{m+1}$$

So the result is true for $\forall n \in \mathbb{N}$.

Now, for proving the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy, we consider two cases.

Case 1: $m + n \leq k + l$

Claim that, for $j \in \mathbb{N}$,

$$d_X(F^{j+1}(x_0, y_0), F^j(x_0, y_0)) \leq (k + l)^j [d_X(x_1, x_0) + d_Y(y_1, y_0)] \quad (3)$$

$$d_Y(G^{j+1}(y_0, x_0), G^j(y_0, x_0)) \leq (k + l)^j [d_Y(y_1, y_0) + d_X(x_1, x_0)] \quad (4)$$

The proof is by mathematical induction.

For $j = 1$, consider

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq k d_X(F(x_0, y_0), x_0) + l d_Y(G(y_0, x_0), y_0) \\ &\leq (k + l) [d_X(x_1, x_0) + d_Y(y_1, y_0)] \end{aligned}$$

Similarly we prove that $d_Y(G^2(y_0, x_0), G(y_0, x_0)) \leq (k + l)[d_X(x_1, x_0) + d_Y(y_1, y_0)]$.

ie, the result is true for $j = 1$.

Now assume that the claim is true for $j \leq t$, and prove for $j = t + 1$.

Consider,

$$\begin{aligned} d_X(F^{t+2}(x_0, y_0), F^{t+1}(x_0, y_0)) &= d_X(F(F^{t+1}(x_0, y_0), G^{t+1}(y_0, x_0)), F(F^t(x_0, y_0), G^t(y_0, x_0))) \\ &\leq k d_X(F^{t+1}(x_0, y_0), F^t(x_0, y_0)) + l d_Y(G^{t+1}(y_0, x_0), G^t(y_0, x_0)) \\ &\leq k(k + l)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] + l (k + l)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\ &= (k + l)^{t+1} [d_X(x_1, x_0) + d_Y(y_1, y_0)] \end{aligned}$$

Similarly we get $d_Y(G^{t+2}(y_0, x_0), G^{t+1}(y_0, x_0)) \leq (k + l)^{t+1} [d_X(x_1, x_0) + d_Y(y_1, y_0)]$.

Thus the claim is true for all $j \in \mathbb{N}$.

Now we prove that $\{F^j(x_0, y_0)\}$ and $\{G^j(y_0, x_0)\}$ are Cauchy sequences in X and Y respectively.

For $t > j$, consider

$$\begin{aligned} d_X(F^j(x_0, y_0), F^t(x_0, y_0)) &\leq d_X(F^j(x_0, y_0), F^{j+1}(x_0, y_0)) + d_X(F^{j+1}(x_0, y_0), F^{j+2}(x_0, y_0)) + \dots \\ &\quad + d_X(F^{t-1}(x_0, y_0), F^t(x_0, y_0)) \\ &\leq \left[(k + l)^j + (k + l)^{j+1} + \dots + (k + l)^{t-1} \right] [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\ &\leq \frac{\delta_1^j}{1 - \delta_1} \left[d_X(x_1, x_0) + d_Y(y_1, y_0) \right] \text{ where } \delta_1 = k + l < 1 \\ &\rightarrow 0 \text{ as } j \rightarrow \infty \end{aligned}$$

ie, $\{F^j(x_0, y_0)\}$ is a Cauchy sequence in X.

Similarly we can prove that $\{G^j(y_0, x_0)\}$ is a Cauchy sequence in Y .

Case 2: $k + l < m + n$.

Now we claim that

$$d_X(F^{j+1}(x_0, y_0), F^j(x_0, y_0)) < (m + n)^j [d_X(x_1, x_0) + d_Y(y_1, y_0)] \quad (5)$$

$$d_Y(G^{j+1}(y_0, x_0), G^j(y_0, x_0)) < (m + n)^j [d_Y(y_1, y_0) + d_X(x_1, x_0)] \quad (6)$$

For $j = 1$

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq k d_X(F(x_0, y_0), x_0) + l d_Y(G(y_0, x_0), y_0) \\ &\leq (k + l) [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\ &< (m + n) [d_X(x_1, x_0) + d_Y(y_1, y_0)] \end{aligned}$$

Similarly we can get $d_Y(G^2(y_0, x_0), G(y_0, x_0)) < (m + n)[d_Y(y_1, y_0) + d_X(x_1, x_0)]$.

Assume that the claim is true for $j \leq t$. Now we prove the claim for $j = t + 1$.

Consider,

$$\begin{aligned} d_X(F^{t+2}(x_0, y_0), F^{t+1}(x_0, y_0)) &= d_X(F(F^{t+1}(x_0, y_0), G^{t+1}(y_0, x_0)), F(F^t(x_0, y_0), G^t(y_0, x_0))) \\ &\leq k d_X(F^{t+1}(x_0, y_0), F^t(x_0, y_0)) + l d_Y(G^{t+1}(y_0, x_0), G^t(y_0, x_0)) \\ &< k(m + n)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] + l (m + n)^t [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\ &< (m + n)^{t+1} [d_X(x_1, x_0) + d_Y(y_1, y_0)] \end{aligned}$$

Similarly we can prove that

$d_Y(G^{t+2}(y_0, x_0), G^{t+1}(y_0, x_0)) < (m + n)^{t+1} [d_Y(y_1, y_0) + d_X(x_1, x_0)]$. Hence the claim is true for all $j \in \mathbb{N}$.

Now we prove that $\{F^j(x_0, y_0)\}$ and $\{G^j(y_0, x_0)\}$ are Cauchy sequences in X and Y respectively.

For $t > j$, consider

$$\begin{aligned} d_X(F^t(x_0, y_0), F^j(x_0, y_0)) &\leq d_X(F^j(x_0, y_0), F^{j+1}(x_0, y_0)) + d_X(F^{j+1}(x_0, y_0), F^{j+2}(x_0, y_0)) + \dots \end{aligned}$$

$$\begin{aligned}
& + d_X(F^{t-1}(x_0, y_0), F^t(x_0, y_0)) \\
& < \left[(m+n)^j + (m+n)^{j+1} + \dots + (m+n)^{t-1} \right] [d_X(x_1, x_0) + d_Y(y_1, y_0)] \\
& \leq \frac{\delta_2^j}{1 - \delta_2} [d_X(x_1, x_0) + d_Y(y_1, y_0)] \text{ where } \delta_2 = m+n < 1 \\
& \rightarrow 0 \text{ as } j \rightarrow \infty
\end{aligned}$$

So, $\{F^j(x_0, y_0)\}$ is a Cauchy sequence in X .

Similarly we can prove that $\{G^j(y_0, x_0)\}$ is a Cauchy sequence in Y .

Since X and Y are complete metric spaces, we have $\lim_{j \rightarrow \infty} F^j(x_0, y_0) = x$ and $\lim_{j \rightarrow \infty} G^j(y_0, x_0) = y$, where $x \in X$ and $y \in Y$.

Now we prove that $F(x, y) = x$ and $G(y, x) = y$.

Consider,

$$\begin{aligned}
d_X(F(x, y), x) &= \lim_{j \rightarrow \infty} d_X(F(F^j(x_0, y_0), G^j(y_0, x_0)), F^j(x_0, y_0)) \\
&= \lim_{j \rightarrow \infty} d_X(F^{j+1}(x_0, y_0), F^j(x_0, y_0)) \\
&= 0
\end{aligned}$$

Therefore $F(x, y) = x$. Similarly we can prove that $G(y, x) = y$. \square

Example 2.1 Let $X = (-\infty, 0]$ and $Y = [0, \infty)$ with usual order and usual metric. Define $F : X \times Y \rightarrow X$ by $F(x, y) = \frac{x}{3} - \frac{y}{4}$ and $G : Y \times X \rightarrow Y$ by $G(y, x) = \frac{y}{8} - \frac{x}{6}$. Then it is easy to check that F satisfies (1) with $k = \frac{1}{3}, l = \frac{1}{4}$ and G satisfies (2) with $m = \frac{1}{8}, n = \frac{1}{6}$ and $(0, 0)$ is the FG-coupled fixed point.

Corollary 2.1 [5, Theorem 2.1] *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X . Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \quad \forall x \geq u, y \leq v. \quad (7)$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof: Taking $X = Y$, $F = G$ and $k = l = m = n = \frac{k}{2}$ in Theorem 2.1 we obtain the result. \square

Remark 2.1 It can be shown that FG- coupled fixed point is unique provided that for every $(x, y), (x^*, y^*) \in X \times Y$, there exist a $(z_1, z_2) \in X \times Y$ that is comparable to (x, y) and (x^*, y^*) . The result is proved in the following theorem.

Theorem 2.2 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two continuous mappings having the mixed monotone property. For every $(x, y), (x^*, y^*) \in X \times Y$ there exist a $(z_1, z_2) \in X \times Y$ that is comparable to (x, y) and (x^*, y^*) . Assume that there exist $k, l, m, n \in [0, 1)$; $k + l < 1$ and $m + n < 1$ with*

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, u) + l d_Y(y, v) \quad \forall x \geq_{P_1} u, y \leq_{P_2} v$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, v) + n d_X(x, u) \quad \forall x \leq_{P_1} u, y \geq_{P_2} v.$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exist a unique FG-coupled fixed point.

Proof: The existence of FG- coupled fixed point is followed by the proof of Theorem 2.1. Now we prove that if (x^*, y^*) is another FG- coupled fixed point then $d((x, y), (x^*, y^*)) = 0$ where $x = \lim_{j \rightarrow \infty} F^j(x_0, y_0)$ and $y = \lim_{j \rightarrow \infty} G^j(y_0, x_0)$.

Case 1: If (x, y) is comparable to (x^*, y^*) with respect to the ordering in $X \times Y$, then $(F^j(x, y), G^j(y, x)) = (x, y)$ is comparable to $(F^j(x^*, y^*), G^j(y^*, x^*)) = (x^*, y^*)$ for every $j = 1, 2, 3, \dots$

If $m + n \leq k + l$, consider,

$$\begin{aligned} d((x, y), (x^*, y^*)) &= d_X(x, x^*) + d_Y(y, y^*) \\ &= d_X(F^j(x, y), F^j(x^*, y^*)) + d_Y(G^j(y, x), G^j(y^*, x^*)) \\ &\leq 2^j(k + l)^j [d_X(x, x^*) + d_Y(y, y^*)] \\ &\rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Which implies that $d((x, y), (x^*, y^*)) = 0$.

Similarly for $k + l < m + n$ we get

$$\begin{aligned} d((x, y), (x^*, y^*)) &< 2^j(m + n)^j[d_X(x, x^*) + d_Y(y, y^*)] \\ &\rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Therefore $d((x, y), (x^*, y^*)) = 0$.

Case 2: If (x, y) is not comparable to (x^*, y^*) , then there exist $(z_1, z_2) \in X \times Y$ such that which is comparable to (x, y) and (x^*, y^*) .

Without loss of generality, consider $m + n \leq k + l$, then

$$\begin{aligned} d((x, y), (x^*, y^*)) &= d((F^j(x, y), G^j(y, x)), (F^j(x^*, y^*), G^j(y^*, x^*))) \\ &\leq d((F^j(x, y), G^j(y, x)), (F^j(z_1, z_2), G^j(z_2, z_1))) + \\ &\quad d((F^j(z_1, z_2), G^j(z_2, z_1)), (F^j(x^*, y^*), G^j(y^*, x^*))) \\ &= d_X(F^j(x, y), F^j(z_1, z_2)) + d_Y(G^j(y, x), G^j(z_2, z_1)) + d_X(F^j(z_1, z_2), F^j(x^*, y^*)) + \\ &\quad d_Y(G^j(z_2, z_1), G^j(y^*, x^*)) \\ &\leq 2^j(k + l)^j\{[d_X(x, z_1) + d_Y(y, z_2)] + [d_X(z_1, x^*) + d_Y(z_2, y^*)]\} \\ &\rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Therefore $d((x, y), (x^*, y^*)) = 0$. \square

We can replace the continuity of F and G by other conditions to get the existence of FG- coupled fixed point, as shown in the following theorem.

Theorem 2.3 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings satisfying the mixed monotone property. Assume that X and Y having the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.*

Also assume that there exist $k, l, m, n \in [0, 1)$ such that $k + l < 1$, $m + n < 1$ with

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, u) + l d_Y(y, v) \quad \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (8)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, v) + n d_X(x, u) \quad \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (9)$$

If there exist $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$, then there exist $(x, y) \in X \times Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in the proof of Theorem 2.1, we get $\lim_{j \rightarrow \infty} F^j(x_0, y_0) = x$ and $\lim_{j \rightarrow \infty} G^j(y_0, x_0) = y$.

Now we have,

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{j+1}(x_0, y_0)) + d_X(F^{j+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^j(x_0, y_0), G^j(y_0, x_0))) + d_X(F^{j+1}(x_0, y_0), x) \\ &\leq k d_X(x, F^j(x_0, y_0)) + l d_Y(y, G^j(y_0, x_0)) + d_X(F^{j+1}(x_0, y_0), x) \\ &\rightarrow 0 \text{ as } j \rightarrow \infty \end{aligned}$$

Therefore $x = F(x, y)$.

Similarly using (9) and $\lim_{j \rightarrow \infty} G^j(y_0, x_0) = y$, we get $y = G(y, x)$. \square

Corollary 2.2 [5, Theorem 2.2] *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property*

- (i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$*
- (ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.*

Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exist $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)] \quad \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof: Taking $X = Y$, $F = G$ and $k = l = m = n = \frac{k}{2}$ in Theorem 2.3, we get the result. \square

Remark 2.2 In Theorem 2.3 if add the condition: for every $(x, y), (x^*, y^*) \in X \times Y$, there exist a $(z_1, z_2) \in X \times Y$ that is comparable to both (x, y) and (x^*, y^*) , we get unique FG- coupled fixed point.

Remark 2.3 If we take $k = l = \frac{a}{2}$ and $m = n = \frac{b}{2}$ where $a, b \in [0, 1)$ with $a + b < 1$ in Theorems 2.1, 2.2 and 2.3, we get Theorems 2.1, 2.2 and 2.3 respectively of [8].

Remark 2.4 If we take $k = m$ and $l = n$ in Theorems 2.1, 2.2 and 2.3, we get Theorems 2.4, 2.5 and 2.6 respectively of [8].

Theorem 2.4 Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist $k, l, m, n \in [0, \frac{1}{2})$ satisfying

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, F(x, y)) + l d_X(u, F(u, v)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (10)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, G(y, x)) + n d_Y(v, G(v, u)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (11)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in Theorem 2.1 we can show that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Using inequalities (10) and (11) we get

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq k d_X(x_n, F(x_n, y_n)) + l d_X(x_{n-1}, F(x_{n-1}, y_{n-1})) \\ &= k d_X(x_n, x_{n+1}) + l d_X(x_{n-1}, x_n) \\ \text{ie, } (1 - k) d_X(x_{n+1}, x_n) &\leq l d_X(x_{n-1}, x_n) \\ \text{ie, } d_X(x_n, x_{n+1}) &\leq \frac{l}{1 - k} d_X(x_{n-1}, x_n) \\ &= \delta_1 d_X(x_{n-1}, x_n) \text{ where } \delta_1 = \frac{l}{1 - k} < 1 \\ &\leq \delta_1^2 d_X(x_{n-2}, x_{n-1}) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & \leq \delta_1^n d_X(x_0, x_1) \end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \leq \delta_2^n d_Y(y_1, y_0)$ where $\delta_2 = \frac{m}{1-n}$

Consider $m > n$

$$\begin{aligned} d_X(x_m, x_n) & \leq d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + \dots + d_X(x_{n+1}, x_n) \\ & \leq \delta_1^{m-1} d_X(x_1, x_0) + \delta_1^{m-2} d_X(x_1, x_0) + \dots + \delta_1^n d_X(x_1, x_0) \\ & = \delta_1^n \left(1 + \delta_1 + \dots + \delta_1^{m-n-1} \right) d_X(x_1, x_0) \\ & = \frac{\delta_1^n}{1 - \delta_1} d_X(x_1, x_0) \\ & \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X.

Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.

Since by the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the proof of Theorem 2.1 by continuity of F and G we can prove that $F(x, y) = x$ and $G(y, x) = y$. \square

By replacing the continuity of F and G by other conditions we obtain the following existence theorems of FG-coupled fixed point.

Theorem 2.5 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) . *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$*

(ii) . *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$*

Also assume that there exist $k, l, m, n \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, F(x, y)) + l d_X(u, F(u, v)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (12)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, G(v, u)) + n d_Y(v, G(y, x)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (13)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in the proof of Theorem 2.4 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Now we have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq k d_X(x, F(x, y)) + l d_X(F^n(x_0, y_0), F(F^n(x_0, y_0), G^n(y_0, x_0))) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \quad (\text{using (12)}) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq k d_X(x, F(x, y))$ as $n \rightarrow \infty$

This holds only when $d_X(F(x, y), x) = 0$. Therefore we get $F(x, y) = x$.

Similarly using (13) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we can prove $y = G(y, x)$. \square

Remark 2.5 If we put $k = m$ and $l = n$ in Theorems 2.4 and 2.5, we get Theorems 2.7 and 2.8 respectively of [8].

Theorem 2.6 Let (X, d_X, \leq_{P_1}) , (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist $k, l, m, n \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, F(u, v)) + l d_X(u, F(x, y)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (14)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, G(v, u)) + n d_Y(v, G(y, x)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (15)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: As in Theorem 2.1 we have $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

We have

$$\begin{aligned}
d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
&\leq k d_X(x_n, F(x_{n-1}, y_{n-1})) + l d_X(x_{n-1}, F(x_n, y_n)) \quad (\text{Using (14)}) \\
&= k d_X(x_n, x_n) + l d_X(x_{n-1}, x_{n+1}) \\
&\leq l [d_X(x_{n-1}, x_n) + d_X(x_n, x_{n+1})] \\
\text{ie, } d_X(x_n, x_{n+1}) &\leq \frac{l}{1-l} d_X(x_{n-1}, x_n) \\
&= \delta_1 d_X(x_{n-1}, x_n) \quad \text{where } \delta_1 = \frac{l}{1-l} < 1 \\
&\leq \delta_1^2 d_X(x_{n-2}, x_{n-1}) \\
&\vdots \\
&\leq \delta_1^n d_X(x_0, x_1)
\end{aligned}$$

Similarly we get $d_Y(y_{n+1}, y_n) \leq \delta_2^n d_Y(y_1, y_0)$ where $\delta_2 = \frac{m}{1-m}$

Now we prove that $\{F^n(x_0, y_0)\}$ and $\{G^n(y_0, x_0)\}$ are Cauchy sequences in X and Y respectively.

For $m > n$,

$$\begin{aligned}
d_X(x_m, x_n) &\leq d_X(x_m, x_{m-1}) + d_X(x_{m-1}, x_{m-2}) + \dots + d_X(x_{n+1}, x_n) \\
&\leq \delta_1^{m-1} d_X(x_1, x_0) + \delta_1^{m-2} d_X(x_1, x_0) + \dots + \delta_1^n d_X(x_1, x_0) \\
&\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_1, x_0) \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Therefore $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X.

Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y.

By the completeness of X and Y, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the proof of Theorem 2.1 we can show that $x = F(x, y)$ and $y = G(y, x)$. \square

Theorem 2.7 Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property

(i) If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \ \forall n$

(ii) If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \ \forall n$.

Also assume that there exist $k, l, m, n \in \left[0, \frac{1}{2}\right)$ satisfying

$$d_X(F(x, y), F(u, v)) \leq k d_X(x, F(u, v)) + l d_X(u, F(x, y)); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (16)$$

$$d_Y(G(y, x), G(v, u)) \leq m d_Y(y, G(v, u)) + n d_Y(v, G(y, x)); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (17)$$

If there exist $x_0 \in X$, $y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X$, $y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in the proof of Theorem 2.6 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Consider

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq k d_X(x, F((F^n(x_0, y_0), G^n(y_0, x_0)))) + l d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \\ &= k d_X(x, F^{n+1}(x_0, y_0)) + l d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq l d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$.

Therefore we get $F(x, y) = x$.

Similarly using (17) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$, we get $y = G(y, x)$. \square

Remark 2.6 If we put $k = m$ and $l = n$ in Theorems 2.6 and 2.7, we get Theorems 2.9 and 2.10 respectively of [8].

Theorem 2.8 Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist a, b, c with $a + b + c < 1$ satisfying

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(x, y)) + b d_X(u, F(u, v)) + c d_X(x, u); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (18)$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(y, x)) + b d_Y(v, G(v, u)) + c d_Y(y, v); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (19)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in Theorem 2.1 we have $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Now we claim that

$$d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{b+c}{1-a}\right)^n d_X(x_0, x_1) \quad (20)$$

$$d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq \left(\frac{a+c}{1-b}\right)^n d_Y(y_0, y_1) \quad (21)$$

The proof is by mathematical induction with the help of (18) and (19).

For $n=1$, consider

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq a d_X(F(x_0, y_0), F^2(x_0, y_0)) + b d_X(x_0, F(x_0, y_0)) + \\ &\quad c d_X(F(x_0, y_0), x_0) \\ \text{ie, } d_X(F^2(x_0, y_0), F(x_0, y_0)) &\leq \frac{b+c}{1-a} d_X(x_0, x_1) \end{aligned}$$

Thus the inequality (20) is true for $n = 1$.

Now assume that (20) is true for $n \leq m$, and check for $n = m + 1$.

Consider,

$$\begin{aligned} d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \\ = d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \end{aligned}$$

$$\begin{aligned}
&\leq a d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)) + b d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \\
&\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\
\text{ie, } d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &\leq \frac{b+c}{1-a} d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \\
&\leq \left(\frac{b+c}{1-a}\right)^{m+1} d_X(x_0, x_1)
\end{aligned}$$

ie, the inequality (20) is true for all $n \in \mathbb{N}$.

Similarly we can prove the inequality (21).

For $m > n$, consider

$$\begin{aligned}
&d_X(F^n(x_0, y_0), F^m(x_0, y_0)) \\
&\leq d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)) + \dots \\
&\quad + d_X(F^{m-1}(x_0, y_0), F^m(x_0, y_0)) \\
&\leq \left[\left(\frac{b+c}{1-a}\right)^n + \left(\frac{b+c}{1-a}\right)^{n+1} + \dots + \left(\frac{b+c}{1-a}\right)^{m-1} \right] d_X(x_0, x_1) \\
&\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_0, x_1) \text{ where } \delta_1 = \frac{b+c}{1-a} < 1 \\
&\rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X .

Similarly by using inequality (21) we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y .

By the completeness of X and Y , there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the proof of Theorem 2.1, using continuity of F and G we can prove that $F(x, y) = x$ and $G(y, x) = y$. \square

Theorem 2.9 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

- (i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$*
- (ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.*

Also assume that there exist a, b, c with $a + b + c < 1$ satisfying

$$d_X(F(x, y), F(u, v)) \leq a d_X(x, F(x, y)) + b d_X(u, F(u, v)) + c d_X(x, u); \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (22)$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(y, x)) + b d_Y(v, G(v, u)) + c d_Y(y, v); \forall x \leq_{P_1} u, y \geq_{P_2} v. \quad (23)$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in the proof of Theorem 2.8 we obtain $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

We have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq a d_X(x, F(x, y)) + b d_X(F^n(x_0, y_0), F(F^n(x_0, y_0), G^n(y_0, x_0))) + \\ &\quad c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= a d_X(x, F(x, y)) + b d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq a d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$.

Therefore $F(x, y) = x$.

Similarly using (23) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we get $y = G(y, x)$. \square

Remark 2.7 If we take $c = 0$ in Theorems 2.8 and 2.9, we get Theorems 2.7 and 2.8 respectively of [8].

Theorem 2.10 Let $(X, d_X, \leq_{P_1}), (Y, d_Y, \leq_{P_2})$ be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist non-negative a, b, c satisfying

$$\begin{aligned} d_X(F(x, y), F(u, v)) &\leq a d_X(x, F(u, v)) + b d_X(u, F(x, y)) + c d_X(x, u); \\ &\forall x \geq_{P_1} u, y \leq_{P_2} v; \quad 2b + c < 1 \end{aligned} \quad (24)$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(v, u)) + b d_Y(v, G(y, x)) + c d_Y(y, v); \quad (25)$$

$$\forall x \leq_{P_1} u, y \geq_{P_2} v; 2a + c < 1$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: As in the proof of Theorem 2.1, it can be proved that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Now we claim that

$$d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{b+c}{1-b}\right)^n d_X(x_0, x_1) \quad (26)$$

$$d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq \left(\frac{a+c}{1-a}\right)^n d_Y(y_0, y_1) \quad (27)$$

We prove the claim by mathematical induction, using (24) and (25).

For $n = 1$, consider

$$\begin{aligned} d_X(F^2(x_0, y_0), F(x_0, y_0)) &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq a d_X(F(x_0, y_0), F(x_0, y_0)) + b d_X(x_0, F^2(x_0, y_0)) + c d_X(F(x_0, y_0), x_0) \\ &\leq b [d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0))] + c d_X(F(x_0, y_0), x_0) \\ \text{ie, } d_X(F^2(x_0, y_0), F(x_0, y_0)) &\leq \frac{b+c}{1-b} d_X(x_0, x_1) \end{aligned}$$

Thus the inequality (26) is true for $n = 1$.

Now assume that (26) is true for $n \leq m$, then check for $n = m + 1$.

Consider,

$$\begin{aligned} d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &= d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\ &\leq a d_X(F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)) + b d_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \\ &\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\ &\leq b [d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0))] \\ &\quad + c d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\ \text{ie, } d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &\leq \frac{b+c}{1-b} d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) \end{aligned}$$

$$\leq \left(\frac{b+c}{1-b}\right)^{m+1} d_X(x_0, x_1)$$

ie, the inequality (26) is true for all $n \in \mathbb{N}$

Similarly we can prove the inequality (27).

For $m > n$, consider

$$\begin{aligned} d_X(F^n(x_0, y_0), F^m(x_0, y_0)) &\leq d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)) + \dots + \\ &\quad d_X(F^{m-1}(x_0, y_0), F^m(x_0, y_0)) \\ &\leq \left[\left(\frac{b+c}{1-b}\right)^n + \left(\frac{b+c}{1-b}\right)^{n+1} + \dots + \left(\frac{b+c}{1-b}\right)^{m-1} \right] d_X(x_0, x_1) \\ &\leq \frac{\delta_1^n}{1-\delta_1} d_X(x_0, x_1); \text{ where } \delta_1 = \frac{b+c}{1-b} < 1 \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X .

Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y .

Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

By continuity of F and G , as in the Theorem 2.1 we can show that $F(x, y) = x$ and $G(y, x) = y$. \square

In the following theorem we replace the continuity by other conditions to obtain FG-coupled fixed point.

Theorem 2.11 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property*

(i) *If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$*

(ii) *If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.*

Also assume that there exist non-negative a, b, c satisfying

$$\begin{aligned} d_X(F(x, y), F(u, v)) &\leq a d_X(x, F(u, v)) + b d_X(u, F(x, y)) + c d_X(x, u); \\ &\quad \forall x \geq_{P_1} u, y \leq_{P_2} v; 2b + c < 1 \end{aligned} \tag{28}$$

$$d_Y(G(y, x), G(v, u)) \leq a d_Y(y, G(v, u)) + b d_Y(v, G(y, x)) + c d_Y(y, v); \quad (29)$$

$$\forall x \leq_{P_1} u, y \geq_{P_2} v; 2a + c < 1$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following as in the proof of Theorem 2.10 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

We have

$$\begin{aligned} d_X(F(x, y), x) &\leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\ &\leq a d_X(x, F(F^n(x_0, y_0), G^n(y_0, x_0))) + b d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\ &= a d_X(x, F^{n+1}(x_0, y_0)) + b d_X(F^n(x_0, y_0), F(x, y)) \\ &\quad + c d_X(x, F^n(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \end{aligned}$$

ie, $d_X(F(x, y), x) \leq b d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$.

Therefore $F(x, y) = x$.

Also by using (29) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$ we can show that $y = G(y, x)$. \square

Remark 2.8 If we take $c = 0$ in Theorems 2.10 and 2.11, we get Theorems 2.9 and 2.10 respectively of [8].

3 FG-coupled fixed point theorems for generalized quasi-contractions

The concept of quasi-contraction was defined by Ciric [3] in 1974. A self mapping T on a metric space X is said to be a quasi-contraction iff there exist a number h , $0 \leq h < 1$, such that

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$. In 1979 K.M. Das and K.V. Naik [4] introduced the concept of quasi-contraction for two mappings. Inspired by this we generalize the concept of quasi-contraction to a mapping on product space and prove the following theorems.

Theorem 3.1 *Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces. Let $F : X \times Y \rightarrow X$ and $G : Y \times X \rightarrow Y$ be two continuous functions having the mixed monotone property. Assume that there exist $k, l \in \left[0, \frac{1}{2}\right)$ such that*

$$d_X(F(x, y), F(u, v)) \leq k M(x, y, u, v); \quad \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (30)$$

$$d_Y(G(y, x), G(v, u)) \leq l N(y, x, v, u); \quad \forall x \leq_{P_1} u, y \geq_{P_2} v, \quad (31)$$

where

$$M(x, y, u, v) = \max \left\{ d_X(x, u), d_X(x, F(x, y)), d_X(x, F(u, v)), d_X(u, F(u, v)), d_X(u, F(x, y)) \right\}$$

$$N(y, x, v, u) = \max \left\{ d_Y(y, v), d_Y(y, G(y, x)), d_Y(y, G(v, u)), d_Y(v, G(v, u)), d_Y(v, G(y, x)) \right\}.$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: As in Theorem 2.1, it can be proved that $\{x_n\}$ is increasing in X and $\{y_n\}$ is decreasing in Y .

Now we claim that

$$d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \left(\frac{k}{1-k}\right)^n d_X(x_0, x_1) \quad (32)$$

$$d_Y(G^{n+1}(y_0, x_0), G^n(y_0, x_0)) \leq \left(\frac{l}{1-l}\right)^n d_Y(y_0, y_1) \quad (33)$$

The proof of the claim is by mathematical induction using (30) and (31).

For $n = 1$, consider

$$\begin{aligned} & d_X(F^2(x_0, y_0), F(x_0, y_0)) \\ &= d_X(F(F(x_0, y_0), G(y_0, x_0)), F(x_0, y_0)) \\ &\leq k M(F(x_0, y_0), G(y_0, x_0), x_0, y_0) \\ &= k \max \left\{ d_X(F(x_0, y_0), x_0), d_X(F(x_0, y_0), F^2(x_0, y_0)), d_X(F(x_0, y_0), F(x_0, y_0)), \right. \\ &\quad \left. d_X(x_0, F(x_0, y_0)), d_X(x_0, F^2(x_0, y_0)) \right\} \end{aligned}$$

$$\begin{aligned}
&= k \max \left\{ d_X(x_0, F(x_0, y_0)), d_X(F(x_0, y_0), F^2(x_0, y_0)), d_X(x_0, F^2(x_0, y_0)) \right\} \\
&\leq k \max \left\{ d_X(x_0, F(x_0, y_0)), d_X(F(x_0, y_0), F^2(x_0, y_0)), \right. \\
&\quad \left. d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0)) \right\} \\
&= k [d_X(x_0, F(x_0, y_0)) + d_X(F(x_0, y_0), F^2(x_0, y_0))] \\
\text{ie, } d_X(F^2(x_0, y_0), F(x_0, y_0)) &\leq \frac{k}{1-k} d_X(x_0, F(x_0, y_0)) \\
&= \frac{k}{1-k} d_X(x_0, x_1)
\end{aligned}$$

So the inequality (32) is true for $n = 1$.

Assume that the result is true for $n \leq m$, then check for $n = m + 1$.

Consider,

$$\begin{aligned}
&d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) \\
&= d_X(F(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0)), F(F^m(x_0, y_0), G^m(y_0, x_0))) \\
&\leq k M(F^{m+1}(x_0, y_0), G^{m+1}(y_0, x_0), F^m(x_0, y_0), G^m(y_0, x_0)) \\
&= k \max \left\{ d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)), d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)), \right. \\
&\quad \left. d_X(F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)), d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)), d_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \right\} \\
&= k \max \left\{ d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)), d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)), \right. \\
&\quad \left. d_X(F^m(x_0, y_0), F^{m+2}(x_0, y_0)) \right\} \\
&\leq k \max \left\{ d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)), d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)), \right. \\
&\quad \left. d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0)) \right\} \\
&= k [d_X(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) + d_X(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0))] \\
\text{ie, } d_X(F^{m+2}(x_0, y_0), F^{m+1}(x_0, y_0)) &\leq \frac{k}{1-k} d_X(F^{m+1}(x_0, y_0), F^m(x_0, y_0)) \\
&\leq \left(\frac{k}{1-k} \right)^{m+1} d_X(x_0, x_1)
\end{aligned}$$

ie, inequality (32) is true for all $n \in \mathbb{N}$.

Similarly we can prove the inequality (33).

Now for $m > n$, consider

$$\begin{aligned}
&d_X(F^m(x_0, y_0), F^n(x_0, y_0)) \\
&\leq d_X(F^m(x_0, y_0), F^{m-1}(x_0, y_0)) + d_X(F^{m-1}(x_0, y_0), F^{m-2}(x_0, y_0)) + \dots \\
&\quad + d_X(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \\
&\leq \left[\left(\frac{k}{1-k} \right)^{m-1} + \left(\frac{k}{1-k} \right)^{m-2} + \dots + \left(\frac{k}{1-k} \right)^n \right] d_X(x_0, x_1)
\end{aligned}$$

$\rightarrow 0$ as $n \rightarrow \infty$

ie, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence in X .

Similarly we can prove that $\{G^n(y_0, x_0)\}$ is a Cauchy sequence in Y .

Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

As in the Theorem 2.1, using the continuity of F and G we can show that $x = F(x, y)$ and $y = G(y, x)$. \square

Example 3.1 Let $X = [-1, 0]$, $Y = [0, 1]$ with usual order and usual metric. Define $F : X \times Y \rightarrow X$ by $F(x, y) = \frac{x}{3}$ and $G : Y \times X \rightarrow Y$ by $G(y, x) = \frac{y}{4}$. Then we can easily check that F satisfies inequality (30) with $k = \frac{1}{3}$ and G satisfies inequality (31) with $l = \frac{1}{4}$ and $(0, 0)$ is the FG- coupled fixed point.

Theorem 3.2 Let (X, d_X, \leq_{P_1}) and (Y, d_Y, \leq_{P_2}) be two partially ordered complete metric spaces and $F : X \times Y \rightarrow X$, $G : Y \times X \rightarrow Y$ be two mappings having the mixed monotone property. Assume that X and Y satisfy the following property

(i) . If a non-decreasing sequence $\{x_n\} \rightarrow x$ then $x_n \leq_{P_1} x \forall n$

(ii) . If a non-increasing sequence $\{y_n\} \rightarrow y$ then $y \leq_{P_2} y_n \forall n$.

Also assume that there exist $k, l \in \left[0, \frac{1}{2}\right)$ such that

$$d_X(F(x, y), F(u, v)) \leq k M(x, y, u, v); \quad \forall x \geq_{P_1} u, y \leq_{P_2} v \quad (34)$$

$$d_Y(G(y, x), G(v, u)) \leq l N(y, x, v, u); \quad \forall x \leq_{P_1} u, y \geq_{P_2} v \quad (35)$$

where

$$M(x, y, u, v) = \max \left\{ d_X(x, u), d_X(x, F(x, y)), d_X(x, F(u, v)), d_X(y, F(u, v)), d_X(u, F(x, y)) \right\}$$

$$N(y, x, v, u) = \max \left\{ d_Y(y, v), d_Y(y, G(y, x)), d_Y(y, G(v, u)), d_Y(v, G(v, u)), d_Y(v, G(y, x)) \right\}.$$

If there exist $x_0 \in X, y_0 \in Y$ satisfying $x_0 \leq_{P_1} F(x_0, y_0)$ and $y_0 \geq_{P_2} G(y_0, x_0)$ then there exist $x \in X, y \in Y$ such that $x = F(x, y)$ and $y = G(y, x)$.

Proof: Following the proof of Theorem 2.7 we get $\lim_{n \rightarrow \infty} F^n(x_0, y_0) = x$ and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$.

Now, consider

$$\begin{aligned}
& d_X(F(x, y), x) \\
& \leq d_X(F(x, y), F^{n+1}(x_0, y_0)) + d_X(F^{n+1}(x_0, y_0), x) \\
& = d_X(F(x, y), F(F^n(x_0, y_0), G^n(y_0, x_0))) + d_X(F^{n+1}(x_0, y_0), x) \\
& \leq k M(x, y, F^n(x_0, y_0), G^n(y_0, x_0)) + d_X(F^{n+1}(x_0, y_0), x) \\
& = k \max \left\{ d_X(x, F^n(x_0, y_0)), d_X(x, F(x, y)), d_X(x, F^{n+1}(x_0, y_0)), \right. \\
& \quad \left. d_X(F^n(x_0, y_0), F^{n+1}(x_0, y_0)), d_X(F^n(x_0, y_0), F(x, y)) \right\} + d_X(F^{n+1}(x_0, y_0), x)
\end{aligned}$$

ie, $d_X(F(x, y), x) \leq k d_X(x, F(x, y))$ as $n \rightarrow \infty$, which implies that $d_X(F(x, y), x) = 0$.
Therefore $F(x, y) = x$.

Also by using inequality (35) and $\lim_{n \rightarrow \infty} G^n(y_0, x_0) = y$, we get $y = G(y, x)$. \square

Remark 3.1 Setting $X = Y$ and $F = G$ in Theorem 2.1 - Theorem 3.2 we get corresponding coupled fixed point theorems in partially ordered complete metric space.

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